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STOCHASTIC MODELING OF DAILY RAINFALL PROCESS IN THE CONTEXT OF CLIMATE CHANGE

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Abstract: Information on extreme rainfall events variations in space and time is essential for the design of various water resources systems. However, it is difficult practically to obtain this information simply based on historical precipitation records due to the random behavior of these phenomena, especially in the climate change context. Hence, statistical and stochastic approaches have been commonly used for describing accurately the spatio-temporal variability of the precipitation process. The objective of this study is to develop a stochastic model to represent the daily precipitation process in the context of climate change. The proposed model (referred herein as MCME- Markov Chain Mixed Exponential) consists of two components: (i) the first component representing the occurrences of daily rainfalls based on the first-order Markov Chain; and (ii) the second component describing the daily rainfall intensities using the Mixed Exponential distribution. In addition, a comparative study was carried out to assess the performance of MCME as compared to the popular LARSWG stochastic model using observed daily precipitation data available at selected rain gauge stations across the province of Quebec, Canada. Both models were calibrated based on the available data for the 1961-1975 period and were validated using the data for the 1976-1990 period. In general, it was found that the proposed MCME could provide comparable or more accurate results than the existing LARSWG in consideration of different graphical and numerical performance criteria for a majority of selected stations. In addition, results of this numerical application have indicated the feasibility, accuracy, and robustness of the MCME.

1 INTRODUCTION

Climate change studies at a local site usually involve daily time series of weather variables to develop future climate scenarios at the site of interest. Global Climate Models (GCMs) have been commonly used to study the present climate and project future climate. However, the spatial resolution of these models is quite coarse preventing an accurate estimation of weather variables at a small scale. Thus, many techniques are developed to link the local weather variables with the large-scale GCM outputs. The most common techniques available include dynamical downscaling (Giorgi and Mearns 1999), artificial neural networks (Gardner and Dorling 1998) and statistical downscaling techniques which in turn include linear regression-based methods (Kilsby et al. 1998) , weather typing classifications (Wilby et al. 2004), and stochastic weather generators (Wilks and Wilby 1999; Semenov and Barrow 1997).

The latter became widely used in the past decade to simulate daily scenarios of future climates at a certain location, necessary for risk assessment, interpolation of missing data and climate change impact assessment (Wilks and Wilby 1999; Semenov and Barrow 1997). This technique is mostly based on perturbing the parameters of weather generators (WGs) with statistics derived from climate scenarios produced by GCMs either on monthly or daily timescales (Wilks 1992; Katz 1996; Semenov and Barrow 1997). The weather generator is calibrated based on observed data from a certain period and the parameters obtained from calibration are then used to generate the synthetic series of any length; thus, the model's outputs represent the climate observed during the period used for calibrating the model. When the parameters of a stochastic weather generator are perturbed, the daily data series generated can be representing a changed climate. While some WGs rely on Markov chains to model rainfall occurrences, ex. WGEN (Richardson and Wright 1984), CLIGEN (Arnold and Elliot 1996), others use semi-empirical distributions (LARS-WG (Semenov and Barrow 1997)). In particular, LARSWG (Long Ashton Research Station Weather Generator) is a popular WG, well-known mostly for its good performance in various climatic conditions (Qian et. al, 2008).

In this study, the MCME model, an occurrence-amount stochastic weather generator is studied. Specifically, the performance of the MCME and LARSWG models is assessed for calibration and validation periods at different rain gauge stations across Quebec, Canada. First, an overview of each model's features is detailed, followed by the description of the data used, the stations locations and the performance evaluation procedure. In the last section, detailed results of the study and discussion are elaborated.

2 OVERVIEW OF THE LARSWG AND MCME MODELS

2.1 LARSWG Stochastic Weather Generator

The LARSWG model is based on the series approach and can generate synthetic daily series of precipitation, maximum and minimum temperatures, and solar radiation (Racsko et al. 1991; Semenov et al. 1998), which can be in turn used to simulate weather data at a given site. After inputting observed daily time-series weather variables at a local site, semi-empirical distributions (SED) are developed monthly for each of the following variables: length of wet series, length of dry series, daily precipitation, minimum and maximum temperatures, solar radiation (Semenov and Barrow 1997). The SED is a cumulative probability distribution function, comprising 23 intervals n, defining empirical cumulative probabilities p_i, from 0 to 1, with i going from 0 to 23, along with the corresponding values of the climatic variables v_i, calculated as follows:

[1] $v_i = min\{v: P(v_{obs} \le v) \ge pi\};\ i = 0, \ldots, n$

where P() represents the probability computed based on the observed data. For the lowest $v_0 = \min\{v_{obs}\}$ and the highest $v_n = \max\{v_{obs}\}$ observed data values, respective fixed probabilities are set where $p_0 = 0$ and $p_n = 1$. For a better approximation of the extreme high daily precipitation values occurring with low probability, low daily precipitation occurring with high probability and extremely long dry and wet series, probability values p are fixed at the corresponding tails of the SED. The synthetic series generation procedure is then based on selecting values from the appropriate distributions using a pseudo-random number generator (Semenov et al. 1998). In particular, to simulate the occurrence of precipitation, alternate wet and dry series are modeled, where the length of the series is randomly selected from the wet or dry SED for a particular month. For a wet day (precipitation > 0 mm), the precipitation amount is generated from the precipitation SED developed for the specified month, independently of the wet series length or the precipitation amount on the previous day. Predicted climate changes can be derived from global or regional climate models and then used to perturb the parameters of distributions to obtain a daily climate scenario at a particular site (Semenov and Stratonovitch 2010). Table 1 gives a summary of the precipitation generation features of the model.

2.2 Markov Chain Mixed Exponential (MCME) Model

The occurrence-amount MCME modeling scheme is a Richardson-type model which consists of two components representing the occurrences and intensities of daily rainfalls. After estimating the parameters of each component, daily rainfall series are simulated through a random generation process. Table 1 gives a summary of the main features of the model.

2.2.1 The Occurrence Process

The first-order Markov Chain is used to model the occurrences of daily rainfall events, as suggested by previous studies, because of its simplicity in estimating the parameters (Chin 1977; Roldan and Woolhiser 1982).The observed rainfall data is inputted and treated as a series of two states (wet or dry), modelled as either 1 or 0 respectively with a first order Markov Chain explaining the dependence between dry and wet days on successive days. Let $X_{t,n}$ be the random variable representing the occurrence and non-occurrence of rainfall on day n of year t:

[2] $X_{t,n} = \begin{cases} 0 & if\ day\ n\ is\ dry \\ 1 & if\ day\ t\ is\ wet \end{cases}$

Accordingly, the transition probabilities of the first-order Markov Chain can be defined:

[3] $p_{ij}(n) = P\{X_{t,n} = j | X_{t,n} = i\}$ *for* $n, t > 1$

where i and j can be 0 (dry) or 1 (wet).

2.2.1.1 Estimating Transition Probability Parameters

To estimate the transition probabilities, the maximum likelihood (ML) estimation method is used. The observed number of transitions $a_{ij,k}(n)$ from state i on day $(n-1)$ to state j on day n in a period k across the full length of data is recorded (Woolhiser and Pegram 1978). In this case, a year is split into $k = 12$ monthly periods. Hence, the probability of a day to be dry given that the previous day was dry, p_{00}, and, the probability of a day to be dry given that the previous day is wet, p_{10} are computed as:

[4] $p_{00,k}(n) = \frac{a_{00,k}(n)}{a_{00,k}(n)+a_{01,k}(n)}$

[5] $p_{10,k}(n) = \frac{a_{10,k}(n)}{a_{10,k}(n)+a_{11,k}(n)}$

2.2.2 The Rainfall Amount

The mixed exponential distribution was found to be the most accurate function to describe the distribution of daily rainfall intensities as compared to other commonly used distributions such as simple exponential, gamma, and Weibull (Roldan and Woolhiser 1982; Wilks and Wilby 1999). The mixed exponential function, fitted to the daily rainfall amounts x greater than $0.1\ mm$ is given by:

[6] $f(x) = \frac{p}{\mu_1} e^{-\frac{x}{\mu_1}} + \frac{1-p}{\mu_2} e^{-\frac{x}{\mu_2}}$

where $x \geq 0.1\ mm, 0 \leq p \leq 1, 0 < \mu_1 < \mu_2,\ \ f(x)$ being the probability density function and p, μ_1, μ_2 the parameters. Mathematically, this function is a superposition of two simple exponential functions with means μ_1 and μ_2, combined by the mixing probability parameter p (Wilks and Wilby 1999).

2.2.2.1 Estimating the mixed exponential parameters

The ML method is used to estimate the parameters of the mixed exponential distribution, where the likelihood function is defined as:

[7] $L(x|p, \mu_1, \mu_2) = \prod_{i=1}^{n} f(x_j|p, \mu_1, \mu_2)$

For simplicity in solving, the log-likelihood function is derived to be:

$$[8]\ l = log\,L = \sum_{j=1}^{N} log\,[\frac{p}{\mu_1} e^{-\frac{x_j}{\mu_1}} + \frac{1-p}{\mu_2} e^{-\frac{x_j}{\mu_2}}]$$

where N is the sample size. Various methods can be used to find the optimal solutions to maximizing this log-likelihood function (Everitt and Hand 1981). The global optimization technique chosen for this study, is the shuffled complex evolution (SCE) algorithm. This method was developed by Duan et. al (1993) and was found able to provide more accurate and more robust results than the local optimization procedures (Peyron and Nguyen 2004).

2.2.3 Seasonal variability of parameters

The MCME has a total of five parameters (two describing the transitional probabilities and three explaining the mixed-exponential distribution), which can be estimated for 12 sets of monthly data. Each monthly parameter set is then fitted to a finite Fourier series (Woolhiser and Pegram 1978), where the parameters change periodically through the 12 months per year, which is the case of weather processes.

Table 1: Precipitation Features of the MCME and LARSWG Models

Weather Variable	MCME	LARSWG
	Precipitation Status	
Definition of wet day	Precipitation > 0.1 mm	Precipitation > 0 mm
Precipitation Occurrence	1st order 2-state Markov Chain Model	Lengths of alternate wet and dry series chosen from a semi-empirical distribution (SED)
	Precipitation Amount	
Daily Distribution	Mixed exponential distribution	Precipitation SED developed for a particular month
Parameters	- ML Estimation Method - SCE technique for global optimization of the ML function - Fourier series fitted for seasonal variation	Separate parameters calculated monthly

3 LOCATIONS OF STATIONS AND DATA USED

I this study, daily precipitation data between 1961 and 1990 from 9 stations across the province of Quebec in Canada is used to assess the performance of the MCME and the LARSWG models. In addition, minimum and maximum temperatures along with solar radiation data series are needed to run the LARSWG model. Thus, the selection of the weather stations was based on the availability of data between 1961 and 1990 for all variables needed (precipitation, solar radiation, minimum and maximum temperatures). Figure 1 shows the graphical locations of the selected weather stations covering the entire Quebec province with the different weather conditions present, and Table 2 provides more details of their characteristics. The daily precipitation observed data series were extracted from the second generation Adjusted Precipitation for Canada dataset (Mekis and Vincent 2011). The minimum and maximum temperatures data were obtained from the second-generation homogenized temperature (Vincent et al. 2012). The solar radiation observed data series were in turn obtained from the Canadian Weather Energy and Engineering Datasets (CWEEDS) (http://climate.weather.gc.ca/prods_servs/engineering_e.html).

Table 2: Summary of the characteristics of the selected weather stations

Station Name	Latitude	Longitude	Elevation (m)
Bagotville	48° 19' 48'' N	71° 0' 0'' W	159
Inukjuak	58° 28' 12'' N	78° 4' 48'' W	25
Kuujjuaq	58° 6' 0'' N	68° 25' 12'' W	39
Lennoxville	45° 22' 12'' N	71° 49' 12'' W	181
Dorval	45° 28' 12'' N	73° 45' 0'' W	73
Normandin	48° 51' 0'' N	72° 31' 48'' W	137
Roberval	48° 31' 12'' N	72° 16' 12'' W	179
Sept-Iles	50° 13' 12'' N	66° 16' 12'' W	53
Val d'Or	48° 2' 60'' N	77° 46' 48'' W	337

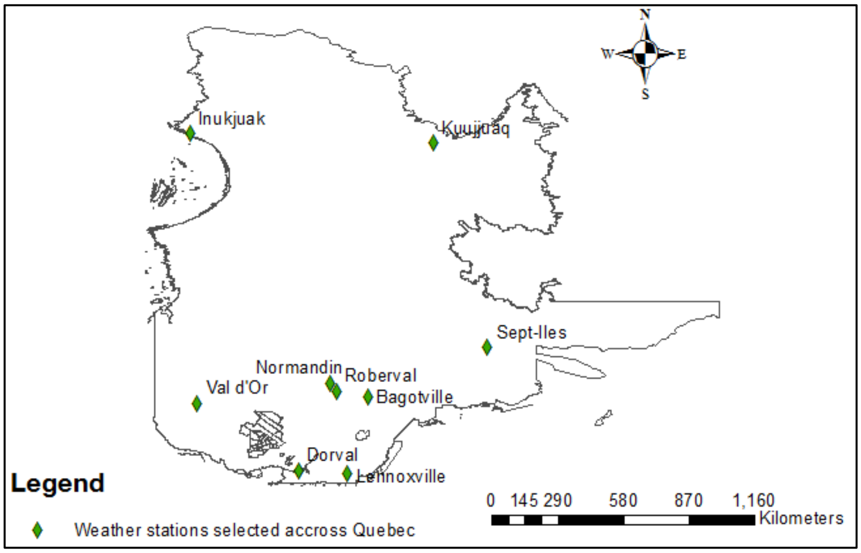

Figure 1: Location of Stations studied in Quebec

4 PERFORMANCE EVALUATION PROCEDURE

The evaluation procedure of each model includes two stages: calibration and validation. The MCME and the LARSWG models take the observed variables series between 1961 and 1975 as input. Then, each model generates 100 precipitation samples of 30-year period, which in turn are split into 2 periods:

1) Calibration period: comparing the 1st 15-year period data with observed rainfall series from 1961 to 1975.

2) Validation period: comparing the 2nd 15-year period data with observed rainfall series from 1976 to 1990.

The performance of each model was then assessed according to Gachon et al. (2005), by studying the indices presented in Table 3, and comparing the results with the observed data. The results are laid out graphically through standard boxplots and numerically through the Root Mean Square Error (RMSE).

[9] $RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(index_{model} - index_{observed})^2}$

The middle band of the displayed boxplots marks the median value, the boxes and whiskers respectively represent the inter-quartile range (IQR) and 1.5x IQR. The red crosses beyond the whiskers denote the outliers. In addition, to further evaluate the performance of each model, a rank sum test was conducted, where the RMSE values, obtained for each index listed in Table 3, are compared between MCME and LARSWG for every station, at calibration and validation stages. The index in the corresponding model with the larger RMSE is assigned a score of 1 while the one with the smaller error gets a score of 0; if the RMSE values of an index for both models are the same, a score of 0 is given to each model. Finally, after summing all scores among all the indices and all the stations, the model with the largest total score is considered less accurate than the other.

Table 3: Evaluation indices and statistics

Index	Unit	Time Scale	Description
Prcp1	%	Season	Percentage of wet days
SDII	mm/day	Season	Precipitation Intensity
CDD	days	Season	Maximum number of consecutive dry days
Wet Spell	days	Season	Maximum number of consecutive wet days
RNPeriod	mm	Season	Greatest 3 days total rainfall
Prec90p	mm	Season	90^{th} percentile of daily precipitation amount
R90N	%	Season	% days with precipitation > 90^{th} percentile calculated for wet days
Means	mm/day	Month	Monthly mean of daily precipitation
Standard Deviation	mm	Month	Monthly standard deviation daily precipitation

5 RESULTS AND DISCUSSION

By going over the calibration and validation results obtained for each station, the performance of each model can be compared by studying the indices listed above. A good performance at the calibration stage indicates that the model is able to replicate the observed data in an accurate way by estimating precisely the corresponding parameters. A good performance at the validation stage shows the ability of the model to predict accurately the observed series in future periods. After comparing the results at different stations, the MCME had a better performance than LARSWG in general. For the 9 stations studied, MCME appeared to be performing better than LARSWG at both the calibration and validation stages, except only for the calibration stage at Lennoxville station, where LARSWG gave better results than MCME. These results show the good ability of MCME to replicate the present-day data and also to predict future rainfall series. Furthermore, the RMSE values obtained for every index are compared between both models by conducting the rank sum test explained previously. Accordingly, table 4 shows the total ranking sum test results obtained for each model at each analysis stage. Tables 5 and 6 displays the total ranking scores at both periods respectively for Dorval and Kuujjuaq stations.

Table 4: Score Ranking Results for MCME and LARSWG at Calibration and Validation Stages

Indices & Statistics	Calibration		Validation	
	MCME	LARSWG	MCME	LARSWG
Prcp1	12	24	13	23
SDII	8	28	10	26
CDD	17	19	14	22
RNPeriod	14	22	15	21
Prec90p	11	25	10	26
R90N	7	29	7	29
Wet Spell	20	16	20	16
Means	37	70	38	69
Standard Deviations	39	68	41	66
Total	**165**	**301**	**168**	**298**

Table 5: Score Ranking Results for MCME and LARSWG - Dorval Station

Indices & Statistics	Calibration		Validation	
	MCME	LARSWG	MCME	LARSWG
Prcp1	1	3	1	3
SDII	0	4	0	4
CDD	0	4	0	4
RNPeriod	2	2	1	3
Prec90p	1	3	0	4
R90N	0	4	0	4
Wet Spell	2	2	2	2
Means	3	9	3	9
Standard Deviations	4	8	5	7
Total	**13**	**39**	**12**	**40**

Table 6: Score Ranking Results for MCME and LARSWG - Kuujuuaq Station

Indices & Statistics	Calibration		Validation	
	MCME	LARSWG	MCME	LARSWG
Prcp1	1	3	1	3
SDII	0	4	1	3
CDD	3	1	1	3
RNPeriod	0	4	1	3
Prec90p	1	3	1	3
R90N	0	4	0	4
Wet Spell	2	2	2	2
Means	3	9	4	8
Standard Deviations	2	10	2	10
Total	**12**	**40**	**13**	**39**

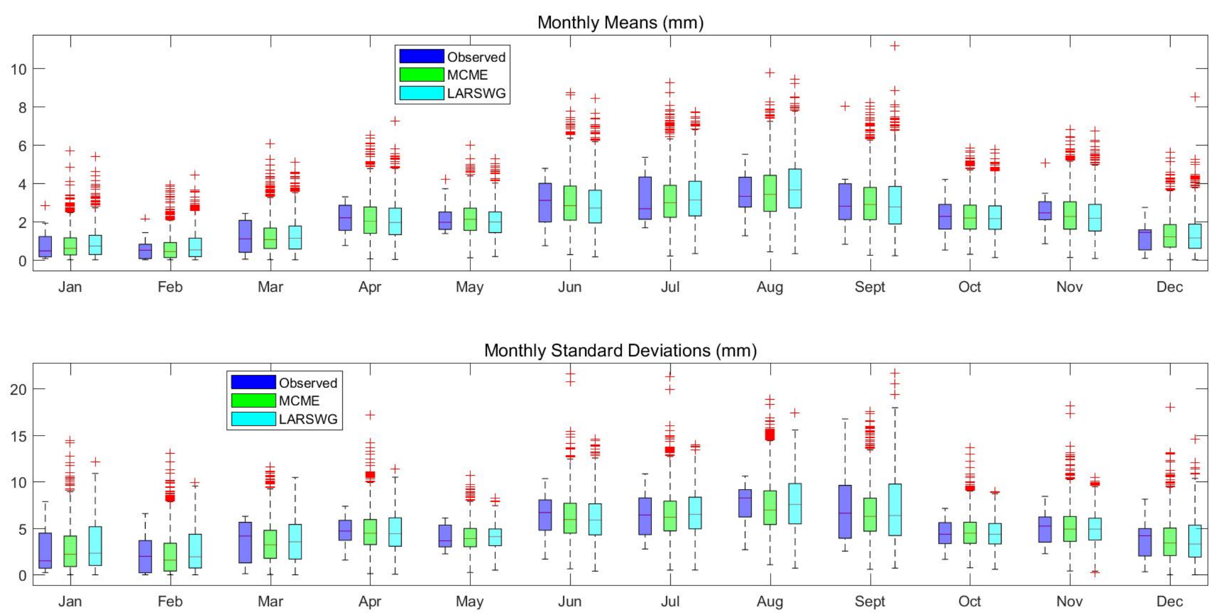

Figure 2: Monthly means and standard deviations - Calibration stage- Dorval station

As shown in Table 4, the rank sum scores obtained with the MCME stochastic model are lower than those found with the LARSWG model for both the calibration and validation periods, indicating lower overall error values and better accuracy for MCME than for LARSWG. For almost all indices calculated, MCME outperformed LARSWG, except for the dry spell (CDD) and the wet spell indices, where the performance of both models was very close, and LARSWG outperformed MCME for the wet spell index. This is explained by the fact that LARSWG relies on the semi-empirical distribution of dry and wet spells in order to simulate synthetic rainfall series; hence, the ability for LARSWG to accurately represent the wet and dry spells in both the calibration and validation periods. However, the stochastic WG still requires 23 parameters in total to generate daily weather variables series, which makes it less flexible for the model to perform accurately in the validation period, contrary to the MCME model. Two stations were chosen to illustrate the graphical results: the first one being Dorval station (Figures 2 and 3), situated South of Quebec, with a relatively wet climate across all seasons, and the second one being Kuujuuaq station (Figures 4 and 5) located in the North of Quebec, and characterized by a totally dry weather during winter (i.e. very low temperatures and no rainfall precipitation, only snow all along the months of December, January and February). As illustrated graphically and represented numerically, MCME outperformed LARSWG in both stations, except for the wet and dry spells where LARSWG showed a better performance.

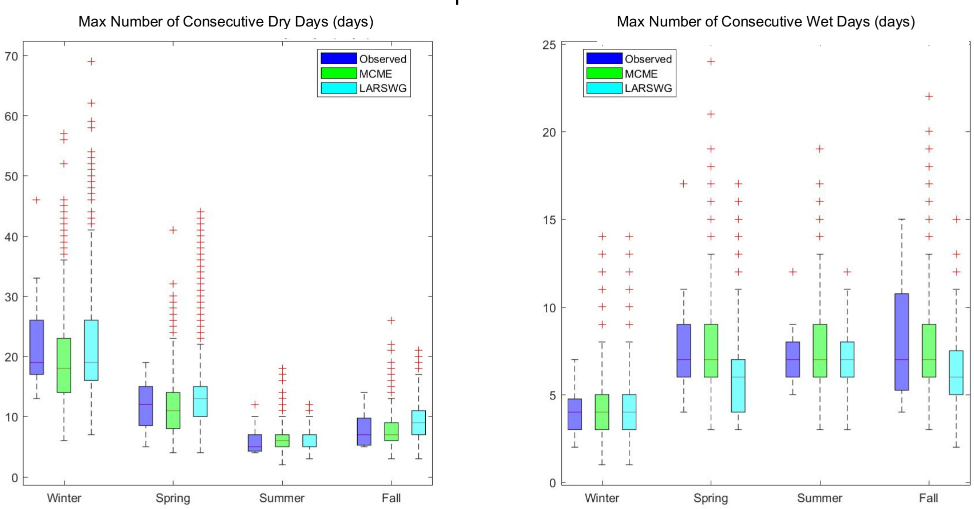

Figure 3: Maximum number of consecutive dry and wet days - Calibration stage- Dorval station

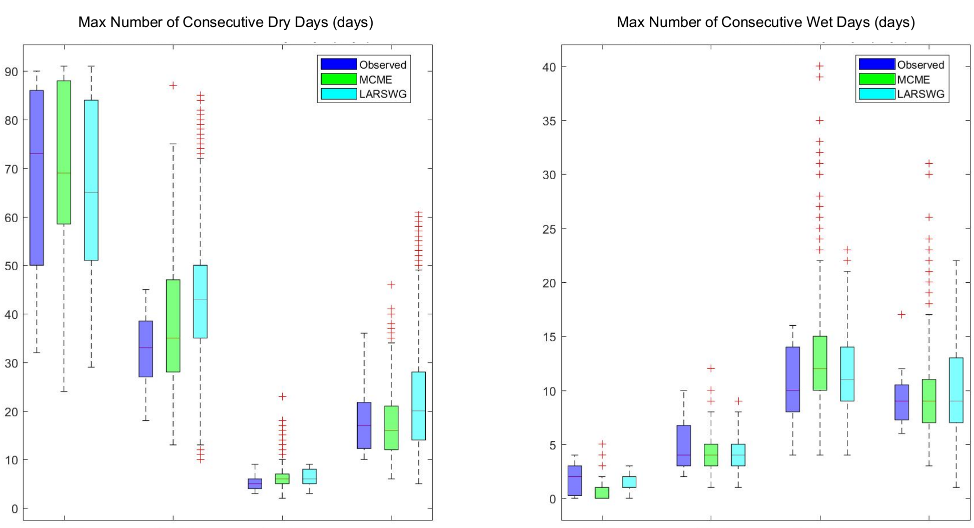

Figure 4: Maximum number of consecutive dry and wet days-validation stage- Kuujjuaq station

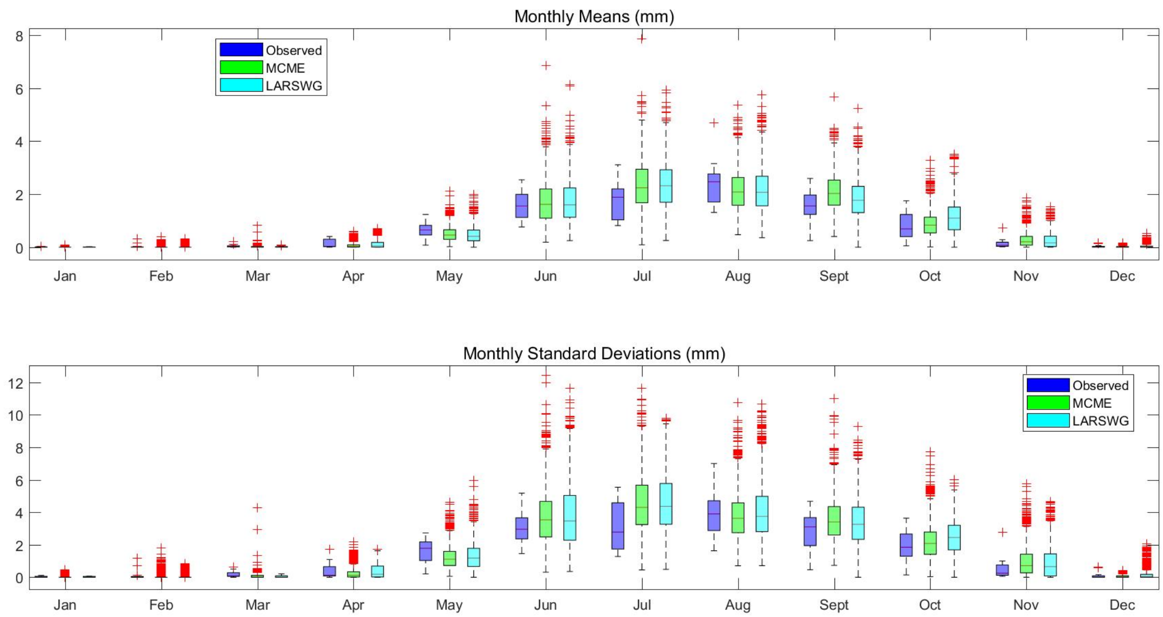

Figure 5: Monthly means and standard deviations- Validation stage- Kuujjuaq station

6 SUMMARY AND CONCLUSIONS

The present study has proposed a stochastic model for representing the daily precipitation process for assessing the climate change impacts on the precipitation a given local site. The proposed model consists of two components: (i) the first component representing the occurrences of daily rainfalls based on the first-order Markov Chain; and (ii) the second component describing the distribution of daily rainfall intensities using the Mixed Exponential distribution. Results of an illustrative application observed daily precipitations from a network of selected stations in Quebec (Canada) have indicated the feasibility and accuracy of the proposed MCME for describing the daily precipitation process. In addition, it was found that the MCME could provide a better performance than the existing LARSWG for a majority of cases considered. Further studies are planned to describe the linkage between the MCME parameters and the climate change factors for current and future climates.

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REFERENCES

Arnold, C.D. and Elliot, W.J. 1996. “CLIGEN Weaher Generator Predictions of Seasonal Wet and Dry Spells in Canada.” *American Society of Agricultural Engineers.* **39**(3): 969–72.

Chin, E.H. 1977. “Modeling Daily Precipitation Occurrence Process With Markov Chain.” *Water Resources Research,* **13**(6): 949–56.

Duan, Q.Y., Gupta, V.K. and Sorooshian, S. 1993. “Shuffled Complex Evolution Approach for Effective and Efficient Global Minimization.” *Journal of Optimization Theory and Applications,* **76**(3): 501–21. doi:10.1007/BF00939380.

Everitt, B.S. and Hand, D.J. 1981. *Finite Mixture Distribution*. *Journal of Chemical Information and Modeling*. **53**, doi:10.1017/CBO9781107415324.004.
Gachon, P., St-Hilaire, A., Ouarda, T., Nguyen, V-T-V, Lin, C., Milton, J., Chaumont, D. 2005. "A First Evaluation of the Strength and Weaknesses of Statistical Downscaling Methods for Simulating Extremes over Various Regions of Eastern Canada." Final report, Sub-component, Climate Change Action Fund (CCAF), Environment Canada, Montréal, Québec, Canada 15.
Gardner, M.W. and Dorling, S.R. 1998. "Artificial Neural Networks (the Multilayer Perceptron) - a Review of Applications in the Atmospheric Sciences." *Atmospheric Environment,* **32**(14–15): 2627–36, doi:10.1016/S1352-2310(97)00447-0.
Giorgi, F. and Mearns, L.O. 1999. "Introduction to Special Section: Regional Climate Modeling Revisited." *Journal of Geophysical Research: Atmospheres,* **104(**D6): 6335–52, doi:10.1029/98JD02072.
Katz, R.W. 1996. "Use of Conditional Stochastic Models to Generate Climate Change Scenarios." *Climatic Change,* **32**: 237–55.
Kilsby, C.G., Cowpertwait, P.S.P., O'Connell, P.E., and Jones, P.D. 1998. "Predicting Rainfall Statistics in England and Wales Using Atmospheric Circulation Variables." *International Journal of Climatology*, **18**(5): 523–39. doi:10.1002/(sici)1097-0088(199804)18:5<523::aid-joc268>3.3.co;2-o.
Mekis, É., and Vincent, L.A. 2011. "An Overview of the Second Generation Adjusted Daily Precipitation Dataset for Trend Analysis in Canada." *Atmosphere - Ocean,* **49**(2): 163–77. doi:10.1080/07055900.2011.583910.
Peyron, N. and Nguyen, V-T-V 2004. "A Systematic Assessment of Global Optimization Methods for Conceptual Hydrologic Models Calibration." *Hydroinformatics*, 953–60.
Qian, B., Gameda, S. and Hayhoe, H. 2008. "Performance of Stochastic Weather Generators LARS-WG and AAFC-WG for Reproducing Daily Extremes of Diverse Canadian Climates." *Climate Research,* **37** (September): 17–33, doi:10.3354/cr00755.
Racsko, P., Szeidl, L., and Semenov, M. 1991. "A Serial Approach to Local Sotchastic Weather Models." *Ecological Modelling,* **57**: 27–41.
Richardson, C. W. and Wright, D.A. 1984. "WGEN : A Model for Generating Daily Weather Variables." *U. S. Department of Agriculture, Agricultural Research Service* ARS-8: **83**. http://soilphysics.okstate.edu/software/cmls/WGEN.pdf.
Roldan, J. and Woolhiser, D.A. 1982. "Stochastic Daily Precipitation Models 1. A Comparison of Occurrence Processes." *Water Resources Bulletin,* **18**(5): 1451–59.
Semenov, M.A., and Barrow, E.M. 1997. "Use Of A Stochastic Weather Generator In The Development Of Climate Change Scenarios." *Climatic Change,* **35**(4): 397–414, doi:10.1023/a:1005342632279.
Semenov, M.A., Brooks, R.J., Barrow, E.M. and Richardson, C.W. 1998. "Comparison of the WGEN and LARS-WG Stochastic Weather Generators for Diverse Climates." *Climate Research*, **10**(2): 95–107. doi:10.3354/cr010095.
Semenov, M.A., and Stratonovitch, P. 2010. "Use of Multi-Model Ensembles from Global Climate Models for Assessment of Climate Change Impacts." *Climate Research.* **41**: 1–14, doi:10.3354/cr00836.
Vincent, L.A., Wang, X.L., Milewska, E.J., Wan, H., Yang, F. and Swail, V. 2012. "A Second Generation of Homogenized Canadian Monthly Surface Air Temperature for Climate Trend Analysis." *Journal of Geophysical Research Atmospheres,* **117**(17): 1–13. doi:10.1029/2012JD017859.
Wilby, R.L., Charles, S.P., Zorita, E., Timbal, B., Whetton, P., and Mearns, L.O. 2004. "Guidelines for Use of Climate Scenarios Developed from Statistical Downscaling Methods." **27**, doi:citeulike-article-id:8861447.
Wilks, D.S., and Wilby, R.L.L 1999. "The Weather Generation Game: A Review of Stochastic Weather Models." *Progress in Physical Geography,* **23**(3): 329–57, doi:10.1177/030913339902300302.
Wilks, D.S. 1992. "Adapting Stochastic Weather Generation Algorithms for Climate Change Studies." *Climatic Change,* **22**(1): 67–84, doi:10.1007/BF00143344.
Woolhiser, D.A., and Pegram, G.G.S. 1978. "Maximum Likelihood Estimation of Fourier Coefficients to Describe Seasonal Variations of Parameters in Stochastic Daily Precipitation Models." *Journal of Applied Meteorology,* **8**: 34–43, doi:10.1016/S0022-1694(99)00088-8.