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A PLEA FOR UNIFIED DEFLECTION CALCULATION OF REINFORCED CONCRETE FLEXURAL MEMBERS

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Abstract: Calculation of deflection for reinforced concrete flexural members is partly empirical in nature and often uses an effective moment of inertia to account for nonlinear behaviour once the member has cracked. In North America, there is presently one approach being used and another being considered for calculation of deflection for steel reinforced concrete, another two for fibre reinforced polymer (FRP) reinforced concrete, and yet another being used in Europe (Eurocode 2). This paper summarizes the different approaches being used to compute deformation, highlights the advantages and disadvantages of each approach, and identifies instances where computed values of deflection may be incorrect. Recommendations are made for a single unified approach able to provide reasonable predictions of deflection regardless of the member (beam, slab or wall) and type (steel and FRP) or strength of reinforcement.

1 INTRODUCTION

Deflection is a serviceability limit state that needs to be satisfied for reinforced concrete structures. Deflection consists of immediate (short-term) deflection that occurs on application of the load and additional long-term deflection from shrinkage and creep of the concrete under sustained loads. Immediate deflection is computed using classical elastic deflection equations, and in most cases long-term deflection is obtained by multiplying the computed short-term deflection value from the sustained loads with a long-term deflection multiplier that depends on the duration of loading and type of reinforcement. Nonlinearity after cracking is accounted for with an effective moment of inertia I_e that varies between the gross (uncracked) moment of inertia I_g and fully cracked moment of inertia I_{cr} depending on the level of loading relative to the cracking load.

This paper is concerned with immediate deflection only. The different approaches presently being used to compute deflection are summarized and compared. Recommendations are made for a single unified approach able to provide reasonable predictions of deflection for different types of members such as beams, slabs, and slender load bearing walls with different types of reinforcement such as steel and FRP.

2 RATIONAL MODEL FOR EFFECTIVE MOMENT OF INERTIA

Bischoff's (2005, 2007) expression for I_e is based on the premise that tension stiffening is measured relative to the I_{cr} response as shown in Fig. 1, where $\beta_{ts}(1 - I_{cr}/I_g)M_{cr}$ represents the tension stiffening component. The tension stiffening factor $\beta_{ts} = M_{cr}/M_a$ after cracking and varies between 1 for full tension stiffening to 0 for no tension stiffening.

The expression for M_a at an assumed value of curvature ϕ_a is given by

[1] $M_a = E_c I_{cr} \phi_a + \beta_{ts}(1 - I_{cr}/I_g) M_{cr}$

where E_c equals the elastic modulus of concrete. Rearranging Eq. [1] gives

[2] $\phi_a = \dfrac{M_a - \beta_{ts}(1 - I_{cr}/I_g)M_{cr}}{E_c I_{cr}} = \dfrac{M_a \left[1 - \beta_{ts}(1 - I_{cr}/I_g)\left(\frac{M_{cr}}{M_a}\right)\right]}{E_c I_{cr}}$

and setting $\phi_a = M_a/E_c I_e$ leads to Bischoff's (2005) original expression for I_e

[3] $I_e = \dfrac{I_{cr}}{1 - \beta_{ts}\frac{M_{cr}}{M_a}\left(1 - \frac{I_{cr}}{I_g}\right)} = \dfrac{I_{cr}}{1 - \left(\frac{M_{cr}}{M_a}\right)^2 \left(1 - \frac{I_{cr}}{I_g}\right)}$

that forms the basis for I_e in ACI 440.1R-15 (ACI 2015) for fibre reinforced polymer (FRP) reinforced concrete and for the I_e expression being proposed for ACI 318-19 for steel reinforced concrete.

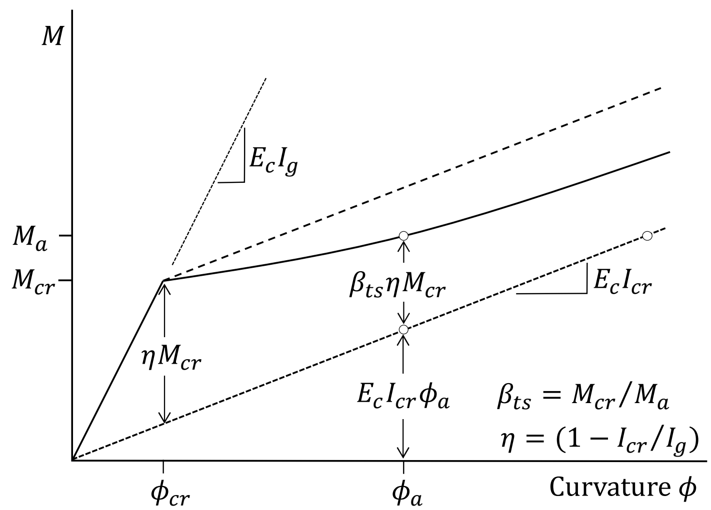


Figure 1: Effect of cracking and tension stiffening on flexural response

3 EFFECTIVE MOMENT OF INERTIA FOR STEEL REINFORCED CONCRETE

For steel reinforced (non-prestressed) concrete, CSA A23.3 (CSA 2014a) uses an effective moment of inertia developed by Branson (1965)

[4] $I_e = I_{cr} + (I_g - I_{cr})\left(\dfrac{M_{cr}}{M_a}\right)^3 \le I_g$

where the cracking moment $M_{cr} = f_r I_g / y_t$, y_t is the distance from the centroidal axis of the gross section (neglecting the reinforcement) to the extreme tension fibre, and M_a equals the maximum moment in the member for the service load being considered. The rupture modulus f_r is taken as one-half the value defined by $0.6\sqrt{f_c'}$ to account for tensile stresses that develop from shrinkage restraint and because of the unconservative nature of using Branson's equation for lightly reinforced slabs (Bartlett 2016). Using one-half the rupture modulus value is equivalent to using $0.5M_{cr}$ in Eq. [4] and f_c' equals the specified concrete compressive strength. Previous editions of A23.3 have used the full cracking moment for beams and one-way slabs, while one-half the rupture modulus value has been used for two-way slabs since 1994 (CSA 1994).

ACI 318 (ACI 2014) uses a rearranged form of Eq. [4]

$$[5] \quad I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \leq I_g$$

but uses the full value of the rupture modulus $f_r = 0.62\sqrt{f_c'}$ for immediate deflection calculation. In other words, the full cracking moment M_{cr} is used.

The I_e expression being proposed for ACI 318-19 given by

$$[6a] \quad I_e = \frac{I_{cr}}{1 - \left(\frac{\frac{2}{3} M_{cr}}{M_a}\right)^2 \left(1 - \frac{I_{cr}}{I_g}\right)}$$

is based on work by Bischoff (2005, 2007) and is applicable for $M_a > (2/3)M_{cr}$. I_e equals I_g for $M_a \leq (2/3)M_{cr}$. The lower cracking moment value of $(2/3)M_{cr}$ accounts for tensile stresses that develop in the concrete from restraint to shrinkage (Scanlon and Bischoff 2008). Bischoff's expression for I_e is also the basis for deflection calculation in the Australian Bridge Design Standard AS 5100.5:2017 and Australian Concrete Structures Standard AS 3600:2018.

Equation [6a] is easily rearranged to give

$$[6b] \quad \frac{1}{I_e} = \left(\frac{\frac{2}{3} M_{cr}}{M_a}\right)^2 \frac{1}{I_g} + \left[1 - \left(\frac{\frac{2}{3} M_{cr}}{M_a}\right)^2\right] \frac{1}{I_{cr}}$$

Eurocode 2 (CEN 2004) takes a weighted average of the uncracked and cracked curvature or deflection values that leads to the following expression for I_e.

$$[7] \quad \frac{1}{I_e} = (1 - \zeta) \frac{1}{I_g} + \zeta \frac{1}{I_{cr}}$$

where $\zeta = 1 - \beta(M_{cr}/M)^2$. The factor β equals 1.0 for single short-term loading and 0.5 for sustained or repeated loading. It is worth mentioning that Eq. [7] with $\beta = 0.5$ is a rearranged form of Eqs. [6] with a reduced cracking moment equal to $0.707M_{cr}$.

Branson's approach (Eqs. [4] and [5]) essentially represents a weighted average of stiffness for two springs in parallel characterized by an uncracked spring with stiffness $E_c I_g$ and a cracked spring with stiffness $E_c I_{cr}$, while Bischoff's approach (Eqs. [3] and [6]) and the approach used by Eurocode 2 (Eq. [7]) represents a weighted average of flexibility corresponding to an uncracked and cracked spring in series. Hence, Branson's computed value of I_e has a bias towards the stiffer uncracked stiffness $E_c I_g$ as the I_g/I_{cr} ratio increases while Bischoff's I_e has a bias towards the more flexible cracked stiffness $E_c I_{cr}$.

4 EFFECTIVE MOMENT OF INERTIA FOR FRP REINFORCED CONCRETE

Serviceability related to deflection and crack control typically governs design of fibre reinforced polymer (FRP) reinforced concrete, and the ultimate flexural design strength is most often more than adequate once serviceability requirements have been satisfied (Veysey and Bischoff 2013, ACI 2015). Hence, deflections must be computed, and a reliable estimate of deflection is needed.

Branson's expression for I_e underestimates deflection of FRP reinforced concrete significantly and numerous attempts have been made since the early 1990's to modify Eq. [5] with correction factors but with limited success (Bischoff et al. 2009). The ACI Guide 440.1R-15 (ACI 2015) for design and construction of FRP reinforced concrete uses the following equation for I_e

$$[8] \quad I_e = \frac{I_{cr}}{1 - \gamma\left(\frac{M_{cr}}{M_a}\right)^2 \left(1 - \frac{I_{cr}}{I_g}\right)}$$

This expression is based on Eq. [3] using the full cracking moment M_{cr}, and includes an integration factor γ to account for changes in stiffness along the member span (Bischoff and Gross 2011a). The factor γ is obtained by integrating curvature along the length of member using Bischoff's expression for I_e (Eq. [3]) as the basis for the cracked section curvature, and depends on the support conditions and type of loading. $\gamma = 1.72 - 0.72(M_{cr}/M_a)$ for a member with a uniformly distributed load. Setting $\gamma = 1$ can be conservatively used for design.

The Canadian Design Standard S806 for FRP reinforced concrete (CSA 2012) requires deflection to be computed by integrating curvature (M/EI) at sections along the member span assuming there is no tension stiffening in the cracked regions. In other words, the curvature equals M/E_cI_g in the uncracked regions where $M \leq M_{cr}$, and M/E_cI_{cr} in the cracked regions where $M > M_{cr}$. Equations for deflection are provided for simple loading cases in lieu of integrating curvature. For example, the midspan deflection Δ of a simply supported member with a span L and central point load P is given by

$$[9] \quad \Delta = \frac{PL^3}{48E_cI_{cr}}\left[1 - 8\left(1 - \frac{I_{cr}}{I_g}\right)\left(\frac{L_g}{L}\right)^3\right]$$

where L_g is the length of the uncracked member in the left and right half of the span. Using Eq. [8] with $\gamma = M_{cr}/M_a$ gives the same value of midspan deflection for a beam with a central point load. For comparison, $\gamma = 3 - 2(M_{cr}/M_a)$ when tension stiffening is included in the cracked regions for the same beam (Bischoff and Gross 2011a). The CSA S806 approach overestimates deflection compared to the ACI 440 approach because tension stiffening is ignored. No guidance is given in the Canadian Highway Bridge Design Code S6-14 (CSA 2014b) for computing deflection of FRP reinforced concrete.

5 EVALUATION OF I_e EXPRESSIONS

Branson's [1965] expression for I_e was calibrated for steel reinforced concrete beams with a reinforcing ratio greater than 1% corresponding to an I_g/I_{cr} ratio less than 3. This expression is not well suited for members with an I_g/I_{cr} ratio greater than 3 and deflection is underestimated when the I_g/I_{cr} ratio exceeds this limit. The greater the I_g/I_{cr} ratio the greater the error. Bischoff and Scanlon (2007) provide a comparison with test results confirming this conclusion.

Figure 2 plots the I_g/I_{cr} ratio for beams and slabs reinforced with either steel or FRP reinforcement, and for slender tilt-up wall panels with a central layer of steel reinforcement. Lightly reinforced members (with $\rho < 1\%$), FRP reinforced concrete slabs and beams, and slender walls all have an I_g/I_{cr} ratio greater than 3. Hence, deflection for these members is underestimated with Branson's original expression for I_e.

Using Branson's expression (Eq. [5]) with a lower cracking moment of $0.75M_{cr}$ compared to Bischoff's expression for I_e at full M_{cr} gives a reasonable estimate of deflection for flexure members with an I_g/I_{cr} ratio up to 6 corresponding to a reinforcing ratio of 0.4%. But deflection will still be underestimated for slabs with a lower reinforcing ratio, and also for slender walls and FRP reinforced concrete.

Figure 3 compares CSA A23.3 using Branson's expression at $0.5M_{cr}$ (Eq. [4]) with the ACI 318 proposed expression at $(2/3)M_{cr}$ (Eq. [6a]) for members with I_g/I_{cr} ratios of 3, 6, 12, and 30. The lower $0.5M_{cr}$ cracking moment used with Branson's equation equals 75% of the $(2/3)M_{cr}$ cracking moment used with Bischoff's expression. Hence, A23.3 works reasonably well (compared with ACI 318) up to an I_g/I_{cr} ratio of about 6 (corresponding to $\rho \cong 0.4\%$), although deflection is now slightly overestimated for members with I_g/I_{cr} less than 3 ($\rho > 1\%$). However, deflection is still underestimated at high I_g/I_{cr} ratios corresponding to slabs with very low reinforcing ratios, slender walls, and FRP reinforced concrete (Bischoff 2018).

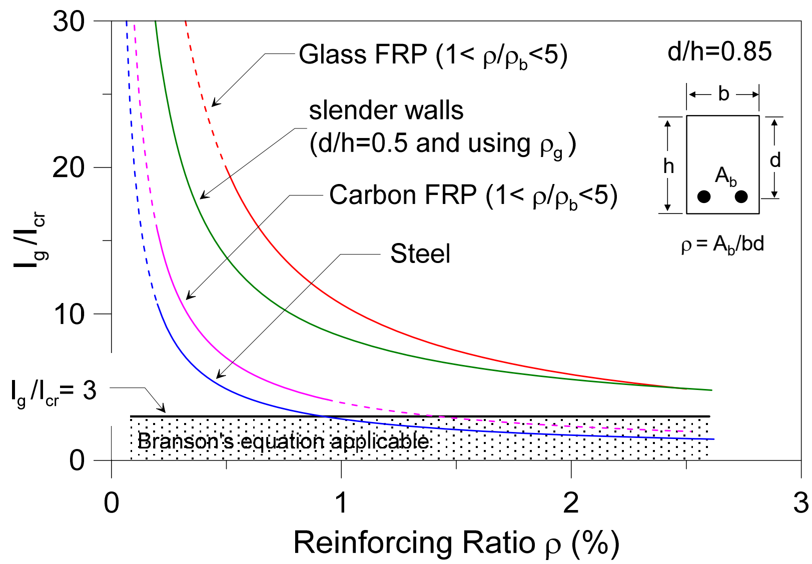


Figure 2: Effect of reinforcement on I_g/I_{cr} ratio

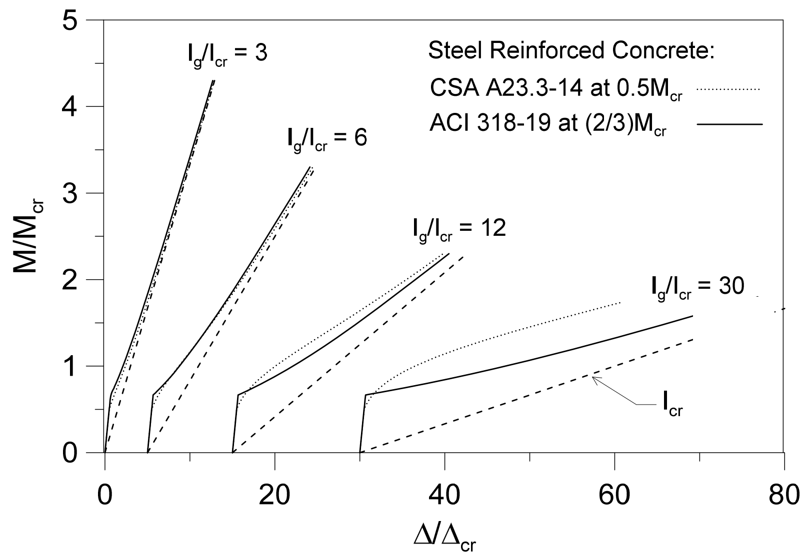


Figure 3: Deflection comparison for steel reinforced concrete

The ACI 440 (Eq. [8]) and CSA S806 approaches for computing deflection of FRP reinforced concrete are compared in Fig. 4 for the response of a simply supported member with a uniformly distributed load. Neither expression is dependent on the I_g/I_{cr} ratio. The CSA S806 response converges rapidly to the I_{cr} response and overestimates deflection considerably compared to ACI 440 since tension stiffening is ignored in the cracked regions of the member (Bischoff and Gross 2011a). Setting $\gamma = 1$ with the ACI 440 expression increases deflection and is conservative.

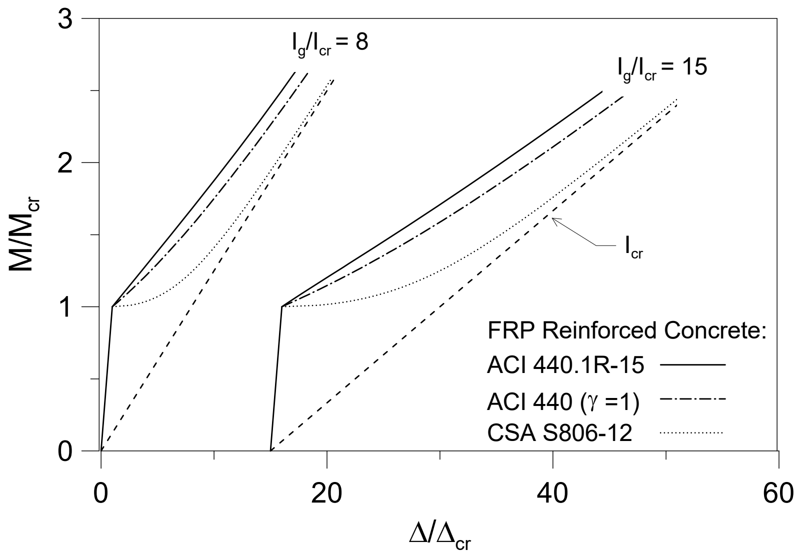


Figure 4: Deflection comparison for FRP reinforced concrete
(simply supported member with a uniformly distributed load)

6 SHRINKAGE RESTRAINT AND AXIAL LOAD

Restraint to shrinkage from either the internal reinforcement, supports or adjacent members induces tensile stresses in the concrete that reduce the cracking moment and increase deflection. This leads to an expression for the reduced cracking moment

$$[10] \quad M'_{cr} = \frac{\left(f_r - f_{res} + \frac{P}{A}\right) I_g}{y_t} = M_{cr}\left(1 - \frac{f_{res}}{f_r} + \frac{P/A}{f_r}\right)$$

that includes the tensile shrinkage restraint stress f_{res} and effect of an axial compressive stress P/A likely to be encountered with load bearing wall panels. For steel reinforced members with no axial load ($P/A = 0$), Scanlon and Bischoff (2008) assume the restraint stress $f_{res} \approx f_r/3$ for lightly reinforced slabs to give $M'_{cr} = (2/3)M_{cr}$. For FRP reinforced slabs and beams, Bischoff and Gross (2011b) recommend using $M'_{cr} = 0.8M_{cr}$ since the shrinkage restraint stress is not as great with FRP reinforcement ($f_{res} \approx f_r/5$). The axial compressive stress for load bearing wall panels acts to offset the tensile restraint stress from shrinkage which can reduce out-of-plane deflection because of the higher cracking moment (Bischoff 2018).

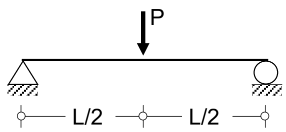
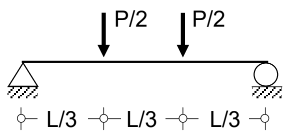
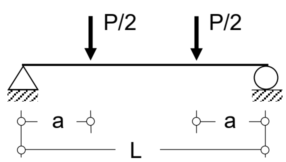
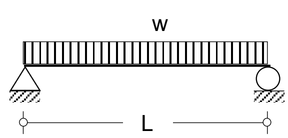
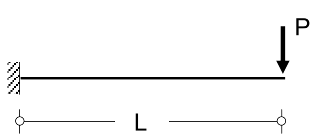
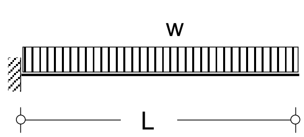
7 UNIFIED APPROACH

Calculation of deflection is recommended using the following general expression for I_e

$$[11] \quad I_e = \frac{I_{cr}}{1 - \gamma\left(\frac{M'_{cr}}{M_a}\right)^2\left(1 - \frac{I_{cr}}{I_g}\right)}$$

where the integration factor γ stiffens the member response by accounting for the variation in stiffness along the member length (accounting for both the stiffer uncracked regions and changes in stiffness in the cracked regions). Table 1 summarizes values of the integration factor used for common types of loading (Bischoff and Gross 2011a). For continuous members, reasonable predictions of deflection are obtained using the effective moment of inertia at midspan computed with the appropriate integration factor from Table 1 for a simply supported member with the same loading conditions (Bischoff 2007, Christie 2014). In lieu of a more refined analysis, it is sufficient and conservative to set $\gamma = 1$.

Table 1 Moment of inertia values for calculating deflection (adapted from Bischoff and Gross 2011a)

Beam & Loading Type	$I_e = \dfrac{I_{cr}}{1-\gamma\left(\frac{M'_{cr}}{M_a}\right)^2\left(1-\frac{I_{cr}}{I_g}\right)}$ for $M_a > M'_{cr}$ $M'_{cr} = 0.67 M_{cr}$ for steel reinforced concrete $M'_{cr} = 0.80 M_{cr}$ for FRP reinforced concrete
	$\Delta = PL^3/48E_cI_e$ and $M_a = PL/4$
	$\gamma = 3 - 2(M'_{cr}/M_a)$
	$\Delta = 23PL^3/1296E_cI_e$ and $M_a = PL/6$
	$\gamma = 1.7 - 0.7(M'_{cr}/M_a)$
	$\Delta = \dfrac{PL^3}{48E_cI_e}[3(a/L) - 4(a/L)^3]$ and $M_a = Pa/2$
	$\gamma = \dfrac{3(a/L) - 4(a/L)^3\xi}{3(a/L) - 4(a/L)^3}$ or $\gamma = (1+\alpha) - \alpha\left(\dfrac{M'_{cr}}{M_a}\right)$ $\xi = 4(M'_{cr}/M_a) - 3$ and $\alpha = \left[\dfrac{4}{0.75(L/a)^2 - 1}\right]$
	$\Delta = 5wL^4/384E_cI_e$ and $M_a = wL^2/8$
	$\gamma \approx 1.72 - 0.72(M'_{cr}/M_a)$ Approximation based on 4-pt loading with a/L = 0.338
	$\Delta = PL^3/3E_cI_e$ and $M_a = PL$
	$\gamma = 3 - 2(M'_{cr}/M_a)$
	$\Delta = wL^4/8E_cI_e$ and $M_a = wL^2/2$
	$\gamma = 1 - 2ln(M'_{cr}/M_a)$

M'_{cr} is defined by Eq. [10] and used to account for tensile stresses from restraint to shrinkage and for compressive stresses from axial loads. Guidance for computing the shrinkage restraint stress is found in Gilbert (1999) and Scanlon and Bischoff (2008). For members with no axial load, reasonable approximations of deflection are obtained using $M'_{cr} = (2/3)M_{cr}$ for steel reinforced concrete and $M'_{cr} = 0.8M_{cr}$ for FRP reinforced concrete.

8 CONCLUSIONS

Despite interest over the past decade to modify and adopt expressions for I_e more capable of providing realistic estimates of deflection, there is still disparity between the different approaches being used in North America and elsewhere for steel and FRP reinforced concrete. A rational approach developed using basic concepts of tension stiffening is proposed for I_e and accounts for shrinkage restraint and axial loads by adjusting the cracking moment. The proposed approach provides reasonable estimates of deflection for both steel and FRP reinforced concrete over a wide range of reinforcing ratios.

Further work is needed to extend this approach to include deflection of cracked prestressed concrete (Bischoff et al. 2018) and the effects of long-term deflection. For long-term deflection, the deflection multipliers used in conjunction with computed values of the immediate short-term deflection under sustained loading require further study and evaluation.

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