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A SIMPLE RIVER MEANDERING MODEL

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ABSTRACT

Principles of mechanics are used to derive a mathematical formulation of river meandering. In its most simple formulation the model generates symmetrical meander loops whose shapes approach those produced by the sine-generated curve. For the more general case the model generates regular upstream-skewed meander loops like those commonly observed in nature. The model proposes explicit equations for evaluating meander characteristics such as loop length or wave length, based on commonly used floodplain and river parameters like the Froude number, transverse slope and the river width.

RÉSUMÉ

Les principes de mécanique sont utilisés pour développer une formulation mathématique des méandres de rivières. Selon la formulation la plus simple le modèle génère des boucles de méandres symétriques. La formulation complète permet de reproduire des boucles méandres asymétriques dirigées vers l'amont. Le modèle permet d'évaluer les caractéristiques de méandres telles que la longueur de la trajectoire, la longueur d'onde et l'amplitude sur la base des caractéristiques physiques de la plaine alluviale et de la rivière elle-même.

Keywords: - Meanders, river, sine-generated curve

1 INTRODUCTION

River meanders have always intrigued scientists. Regular-shaped meander loops are often observed but a succession of regular loops is less frequent. Most of the times the pattern is broken after a few loops only. Changes in shape or in direction occur suddenly without evident physical reason. These irregularities seem to imply that meander loops are sensitive to any modification in the physical properties of the floodplain in the flowing river direction. It is clear that a change in the composition of the material composing the river banks will most certainly disturb the meandering pattern. The fact that regularly shaped meanders are more frequently observed in small rather than large rivers indicates that the variation in surface elevation of the flood plain also plays a major role.

The first serious attempt to mathematically represent river meanders is due to Langbein and Leopold (1966) who have proposed the sine-generated curve, which reproduces fairly well the fatness or roundness of the meander loops observed in nature. However, the sine-generated curve gives no information on the characteristics of the river. Since then, more involved river meander models have been proposed by various researchers (Ikeda *et al.* 1981; Parker *et al.* 1982; Odgaard 1989). These models generally combine dynamic equations for flow in bends with a kinematical description of bank erosion to describe channel migration. Parker *et al.* (1983) have developed a model reproducing the upstream skew commonly encountered in meandering streams.

The regularly-shaped free meander model development presented herein is based on the equations of mechanics and does not require any parameter adjustment. The model allows the derivation of simple explicit equations for meander loop length, meander wave length and meander amplitude.

1. MODEL ASSUMPTIONS

Fig. 1 shows a plan view of a meandering river assumed to be in a quasi-equilibrium state. This condition means that the amplitude and shape of the meanders change slowly with time and can be considered constant during a given time frame (from a few days to several months). The x-axis coincides with the general longitudinal down valley direction.

A regularly-shaped meander is a meander developed in ideal conditions and these ideal conditions are defined by the following set of assumptions:
a) The plan form geometry is defined by the bankfull discharge

b) The river width is constant

c) The water depth is constant

d) The river banks are erodible

e) The longitudinal valley slope is constant

f) The flood plain is gently and uniformly sloping on both sides toward the main longitudinal river channel axis

The transverse flood plain slopes allow surface runoff to drain toward the river channel. This last assumption has rarely, if ever, been considered in meandering models. Nevertheless, it is considered to be an important condition for developing meanders because it forces the river to flow within a limited flood plain band on each side of the main longitudinal river axis. In order to generate symmetrical meanders, it is further assumed that this transverse slope is constant and is the same on both sides of the longitudinal axis. Unequal transverse slopes are possible but would result in asymmetrical meander loops.

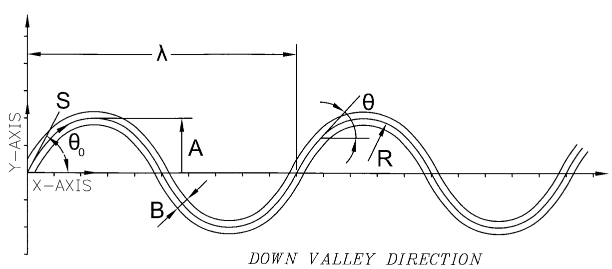

Figure 1. Definition of variables

2 MODEL DEVELOPMENT – BASE CASE

The model development first addresses the simple base case in which it is assumed that the down valley slope, denoted S_X, is much smaller than the transverse flood plain slope, denoted S_Y or, expressed differently, $S_X << S_Y$.

Let us consider an infinitesimal river segment of length Δs. Using the definition of variables shown in Fig. 2, the normal force exerted on the river bank by the curved river segment can be derived using the momentum equation.

[1] $2\rho Q V \sin\left(\frac{\Delta\theta}{2}\right) = w\,\Delta s$

For infinitesimally small Δ*s* the equation becomes

[2] $\frac{\gamma}{g} Q V \frac{d\theta}{ds} = w$

in which γ = unit weight of water; g = acceleration due to gravity; Q = discharge; V = flow velocity; θ = deflection angle relative to x axis; s = distance along river path; and w = force per unit length applied in the river bend.

In order to achieve quasi-equilibrium this force must be resisted by the river banks. The total force exerted by the river banks in a direction normal to the flow is equal to the force on the outer bank minus the force on the inner bank of the bend. Due to the curved flow, a superelevation of the water surface is observed on the outer bank which is counterbalanced by an equivalent drop on the inner bank, as shown in Fig. 3. Let h be the average water depth, assumed to be uniform over a meander loop length. Neglecting secondary currents, the total water depth will be h + Δh on the outer river bank and *h – Δh* on the inner bank. The resulting force per unit length on the segment of length *ds* is then

[3] $w = \frac{1}{2}\gamma (h + \Delta h)^2 - \frac{1}{2}\gamma (h - \Delta h)^2 = 2\gamma h \Delta h$

in which it is assumed that Δ*h* << *h*. Combining Eqs. (2) and (3) it follows that

[4] $\frac{\gamma}{g} Q V \frac{d\theta}{ds} = 2\gamma h \Delta h$

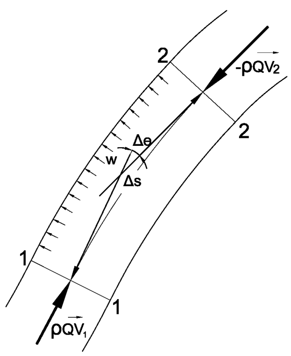

Figure 2. Momentum equation applied to a river bend

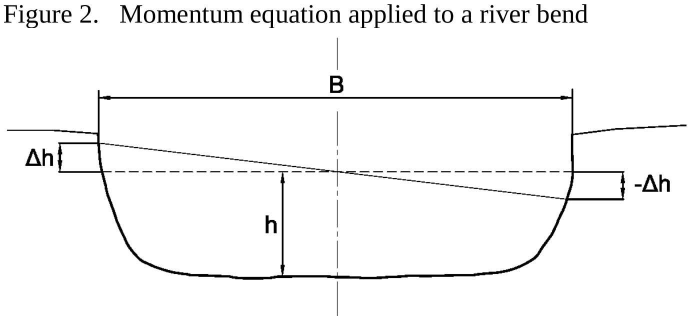

Figure 3. Superelevation of water surface in a river bend

Considering that Q = V B h, where B is the river width, and that dθ/ds is the inverse of the radius of curvature (R), Eq. (4) is equivalent to the well-known expression for estimating the superelevation of the water surface in the outer side of a river bend (Chow 1959).

[5] $\Delta h = \frac{V^2 B}{2 g R}$

Let us now make use of the assumption according to which the floodplain is not flat but is rather gently sloping, on both sides, towards the longitudinal axis, as shown in Fig. 4. The transverse floodplain slope is denoted S_Y.

Assuming a constant transverse slope, the point in the floodplain where the river bank height is exactly equal to the river depth plus the superelevation is at distance *y* from the x-axis.

It follows that Eq. (5) can be rewritten as:

$$[6] \quad \frac{\gamma}{g} Q V \frac{d\theta}{ds} = 2\gamma h \Delta h = -2\gamma h S_Y y$$

in which y is the distance from any given point of the river path to the x-axis. The minus sign is needed because, as shown in Fig. 1, the value of dθ/ds is negative when *y* is positive and vice versa.

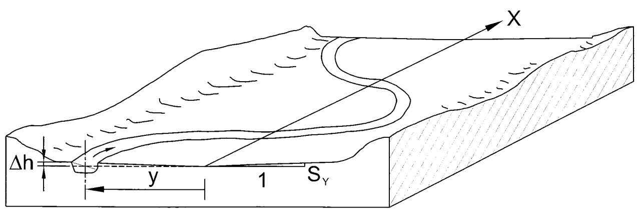

Figure 4. Definition of floodplain transverse slope

Distances *x* and *y* can be expressed in curvilinear coordinates as follows:

$$[7] \quad \begin{aligned} x &= \int \cos\theta \, ds \\ y &= \int \sin\theta \, ds \end{aligned}$$

We can therefore write

$$[8] \quad \frac{\gamma}{g} Q V \frac{d\theta}{ds} = -2\gamma h S_Y y = -2\gamma h S_Y \int \sin\theta \, ds$$

or, after differentiating with respect to *s* and rearranging,

$$[9] \quad \frac{\gamma}{g} Q V \frac{d^2\theta}{ds^2} + 2\gamma h S_Y \sin\theta = 0$$

Equation 9 is similar to other well-known equations in engineering mechanics. For example, the motion of a large amplitude pendulum is described by an equation similar to Eq. 9. The buckling of a column beam subject to large deformations is also defined by an equation of the same form as Eq. 9.

Solving Eq. 9 requires some mathematical treatment which the interested reader can find in Timoshenko and Gere (2009). After some manipulations and changes of variables the trajectory, s, of the river can be expressed in terms of elliptic integrals as follows:

$$[10] \quad s = \mu \left[F(k) - F(k/\varphi) \right]$$

where F(k) and F(k/φ) are respectively the complete and incomplete elliptic integral of the first kind, k = sin (θ_0/2) and

$$[11] \quad \varphi = \sin^{-1}\left(\frac{\sin(\theta/2)}{\sin(\theta_0/2)} \right)$$

F(k) is equivalent to F(k/(π/2)). The constant μ is equal to

[12] $\mu = \sqrt{\frac{QV}{2ghS_Y}} = 4F_rB\sqrt{\frac{1}{2\xi S_Y}}$

where F_r is the Froude number and ξ is the ratio B/h. Parameter μ has the dimension of a length and is characteristic of the meander size. It will hereafter be called the meander scale parameter.

Eq. (10) can be solved numerically (Abramowitz and Stegun 1980). For one quarter period $\varphi = 0$ and for a full period the total meander path length is

[13] $M_L = 4\mu F(k)$

It is also possible to derive mathematical expressions for both x and y which can then be used to derive some characteristics of the meander plan form.

From Eqs. (7) and (10), it can be shown that :

[14] $x = 2\mu[E(k) - E(k/\varphi)] - \mu[F(k) - F(k/\varphi)]$

where E(k) and E(k/φ) are the complete and incomplete elliptic integrals of the second kind and E(k) = E(k/(π/2)). For a full period, *x* corresponds to the meander wave length, λ, and is given by

[15] $\lambda = 4\mu[2E(k) - F(k)]$

Finally, the amplitude of the meander – half the meander width - is estimated using Eqs. (7) and (10) which results in

[16] $y = 2\mu\sqrt{\sin^2\left(\theta_0/2\right) - \sin^2\left(\theta/2\right)}$

and

[17] $A = y(\theta = 0) = 2\mu k$

3 CHARACTERISTICS OF THE MEANDER PATTERNS

Eqs. (10) and (14) can be solved numerically and, in combination with Eq. (16), can be used to draw the meander plan form. Fig 5 illustrates some dimensionless meander patterns for different values of the maximum deflection angle, θ_0. The coordinates are made dimensionless by dividing both x and y by the meander scale parameter (μ). The curves actually represent the channel centerlines of the meandering streams. It can be seen that the pattern increasingly departs from a sine curve as the maximum deflection angle increases. Theoretically, the maximum value of the starting deflection angle for the centerline of the stream cannot exceed about 117.5° otherwise the meander loops come into contact with each other and a cutoff is formed which destroys the meandering pattern. In practice, the maximum deflection angle would be slightly less than 117.5° because of the finite river width.

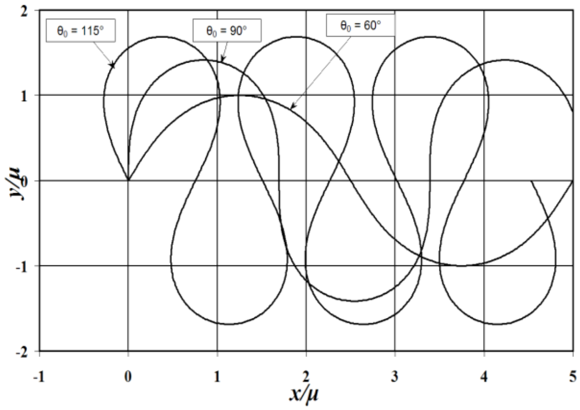

Figure 5. Some typical river meander patterns

It is interesting to visualize how parameter k and the complete elliptic integrals vary with the maximum deflection angle. Fig. 6 shows the variation of parameters k, F(k) and E(k), obtained by numerical integration, as a function of the maximum deflection angle, θ_0. It can be seen that F(k) varies from about 1.57 ($\theta_0 = 0^o$) to 2.13 ($\theta_0 = 118^o$) and, for the same value of parameter μ, the meander path length (Eq. 13) can vary by a factor of 1.36 at most.

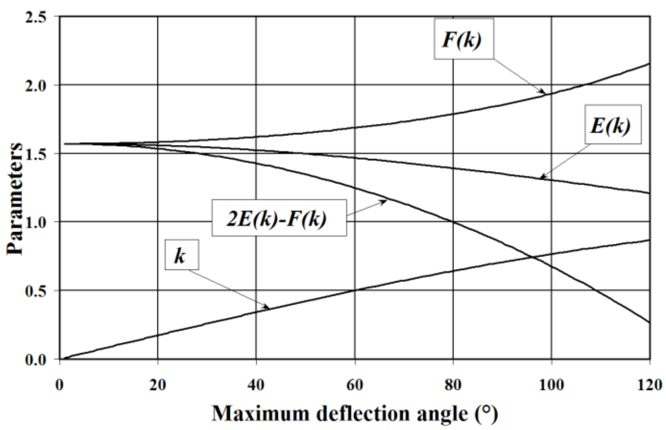

Figure 6. Variation of some general parameter as a function of the maximum deflection angle

Some empirical relationships have been proposed for expressing the wave length as a function of the river width. All these equations have the form $\lambda = \alpha\ B^{\beta}$, where α and β are constants. Different authors have proposed different values for α which vary from 6 to 12 and β is generally close to 1.0. However, Eq. 15 shows that the wave length is not only a function of the river width, *B*, but also depends on the Froude number, the flood plain transverse slope and the maximum deflection angle. For the same value of the length parameter, μ, figure 5 shows, for example, that the meander wave length for a deflection angle of 115º is about 3.5 times less than when the deflection angle is 60º. That ratio decreases as the deflection angle approaches 0º. Figure 7 shows the variation of the meander wave length as a function of the maximum deflection angle for a constant value of the length parameter. One must remember that the maximum deflection angle theoretically does not exceed 117.5º.

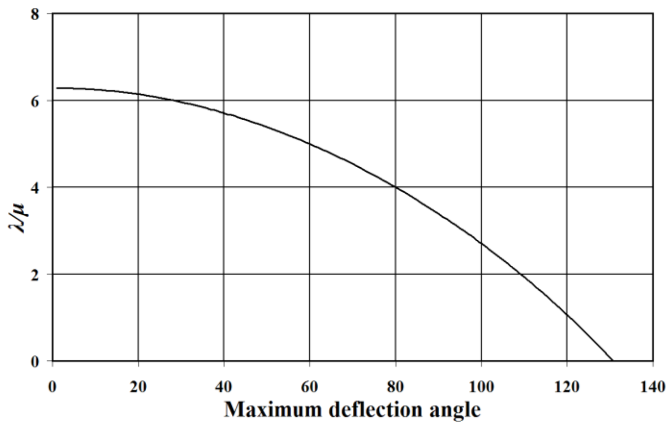

Figure 7. Variation of meander wave length as a function of the maximum deflection angle

The sinuosity, σ , of a meander is defined as the ratio of the path length to the wave length ($\sigma = M_L/\lambda$). Using Eqs. (13) and (15), we can write

[18] $$\sigma = M_L/\lambda = \frac{F(k)}{2E(k) - F(k)}$$

from which it appears that the sinuosity depends solely on parameter k which is a function of the maximum deflection angle. Fig. 8 shows the variation of the sinuosity.

Hey (1976) and Hey and Thorne (1986) recognized that the maximum deflection angle had an influence on the wave length and they proposed to express the full path length as a function of the river width. The equation proposed is

[19] $$M_L = 4\pi B$$

but Eq. (13) can also be written as

[20] $$M_L = 4 F_r B \sqrt{\frac{1}{2\xi S_Y}} F(k)$$

which shows that M_L is still a function of the Froude number, the transverse flood plain slope and the maximum deflection angle. However, the range of variation of factor F(k) (from 1.57 to 2.12) is much smaller than the range of variation of factor 2E(k)-F(k) (from 1.57 to 0.34), which is associated with the meander wave length as shown in Fig. 6. This explains why estimates of meander path length is generally more accurate than estimates of meander wave length. Eqs. (19) and (20) can be used to obtain rough estimates of the floodplain transverse slope. By taking a value of the Froude number equal to 0.50 and an average value of the factor F(k) equal to 1.8, the product ξS_Y is found to be approximately equal to 0.040. The factor ξ, which is equal to the ratio B/h, normally varies, for meandering rivers, between 10 and about 60 (Odgaard 1989). This shows that the flood plain transverse slope can then be expected to vary between about 0.0040 for narrow rivers and 0.0007 for wide rivers.

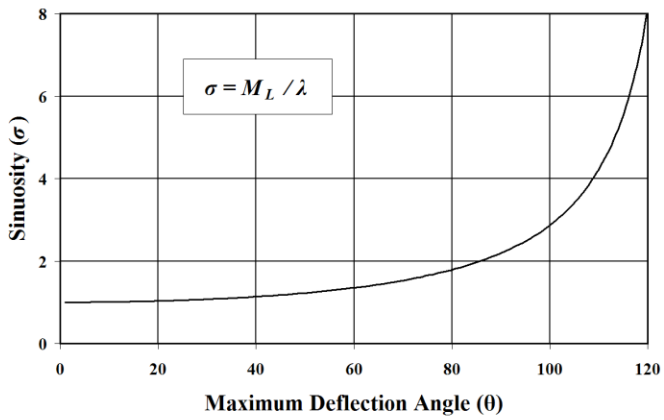

Figure 8. Variation of river meander sinuosity with the maximum deflection angle

4 MODEL DEVELOPMENT – GENERAL CASE

Upstream-skewed meander loops are frequently met in nature as can be seen in Fig. 10. The following paragraphs show the meandering model presented above can also generate upstream-skewed meander loops.

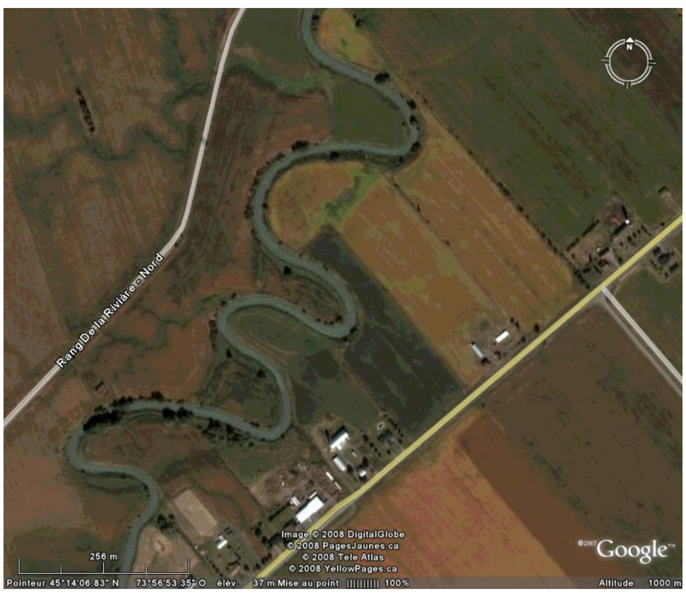

Figure 10. Meander loops in the St. Louis River near Beauharnois (Quebec) (Water flowing from southwest to northeast)

The model developed above generates symmetrical meanders with respect to the down valley axis. This was obtained because it was assumed that the longitudinal valley slope was much smaller than the floodplain transverse slope. The following development addresses the case where the longitudinal slope is not negligible with respect to the transverse slope.

Starting with Eq. (4), which remains valid for the general case, the point in the flood plain whose elevation matches the superelevation caused by the curvature of the flow is defined, according to Fig. 9, by the expression $S_Y\, y - S_X\, x$ and Eq. (6) now becomes:

[21] $$\frac{\gamma}{g} Q V \frac{d\theta}{ds} = 2\gamma h \Delta h = 2\gamma h (\Delta h_Y + \Delta h_X) = -2\gamma h (S_Y\, y - S_X\, x)$$

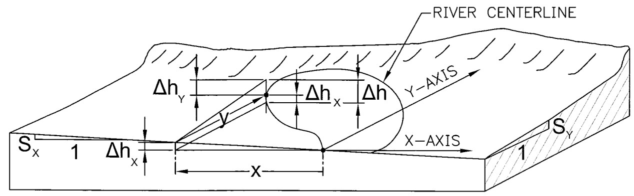

Figure 9. Mathematical representation for the general case

After a few transformations Eq. (21) becomes

[22] $$\frac{\gamma}{g} Q V \frac{d^2(\theta - \delta)}{ds^2} = -2\gamma h Z \sin(\theta - \delta)$$

where $Z = (S_X^2 + S_Y^2)^{1/2}$ and $\delta = \tan^{-1}(S_X / S_Y)$.

Since δ is a constant, Eq. (22) can be rewritten

[23] $$\frac{\gamma}{g} Q V \frac{d^2\Theta}{ds^2} + 2\gamma h Z \sin\Theta = 0$$

where $\Theta = \theta - \delta$. Eq. (23) has the same form as Eq. (9) and will therefore produce similar solutions. The only difference is that Eq. (23) generates meander loops making an angle δ with respect to the x-axis. A representation of the tilted loops is shown in Fig. 11 where the tilting is due to the introduction of the angle δ.

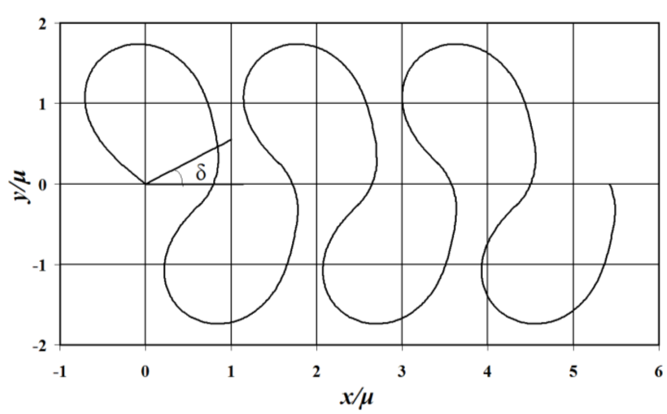

Figure 11. Meander pattern showing upstream skewing

Figure 11 shows the new meander patterns obtained when δ = 30°, which is equivalent to assuming that the ratio of the longitudinal slope to the flood plain transverse slope is about 0.58. The larger the ratio of the slopes the more tilted the patterns. The above development demonstrates that the commonly observed upstream-skewed meander loops result from a non-negligible ratio of the longitudinal slope to the transverse slope.

5 SINE-GENERATED CURVE

Langbein and Leopold (1966) proposed the theory of minimum variance to describe regular meander patterns. They proposed the following equation, also called the sine-generated curve,

[24] $$\theta = \theta_0 \sin 2\pi \frac{s}{M_L}$$

where M_L is the total path length along a full meander cycle. The equation includes a scale parameter, M_L, and a shape parameter, θ_0. By selecting appropriate values for these parameters it is possible to represent the shape of real meanders, including the relatively round curve shown by meanders of large amplitude. However, Eq. 24 provides no information about the variables characterizing the river (discharge, river width, river depth, etc.). The same remark applies to the meander models proposed by Ikeda *et al*. (1981), Parker *et al*. (1982 and 1983) and Odgaard (1989).

As Langbein and Leopold explained, the sine-generated curve is actually an approximation to an equation developed by von Schelling (1951) to describe the most probable path taken by a particle moving between two fixed points in a plane when the path length is fixed. According to von Schelling, the problem of the most frequent path is equivalent to finding the curve which minimizes the curvature along the loop, or

[25] $$\int \left(\frac{d\theta}{ds} \right)^2 ds = \text{a minimum}$$

It is remarkable that the solutions to Eq. 25 found by von Schelling for the path length and for the x coordinate also involve elliptic integrals, similar to Eqs. 10 and 15. His solution for the y coordinate is also similar to Eq. 16. This is a further argument showing that the present model generates meanders loops having the minimum variance.

6 DISCUSSION

The introduction of constant floodplain transverse slope in this model is crucial for the development of regular-shaped minimum variance meander loops. In the simpler form of the model, this condition ensures that the flow curvature is directly proportional to the distance from the main longitudinal river axis (Eq. 6). The assumption is equivalent to stating that the rise in flood plain elevation due to the transverse slope must match the superelevation of the river flow in bends at bankfull conditions. Although it may appear surprising that this condition suffices to generate regular meander loops, it actually ensures that the rate of bank erosion is proportional to the flow curvature. In effect, when the floodplain transverse slope is constant, the difference in level between the water surface in the outer side of a river curve and the flood plain is constant over the whole meander loop. All other conditions being the same, it may then be inferred that the rate of bank erosion is proportional to the flow curvature. However, a discontinuity in the flood plain transverse slope such as a depression will cause the river flow to inundate part of the flood plain, disturbing the flow pattern in the river bank near the depression. This disturbance will be sufficient to break the meander pattern.

Irregularities in the flood plain transverse slope are much more likely to be encountered in very wide flood plains and this may explain why large rivers, although they may be sinuous or braided, rarely exhibit regularly shaped meander loops. Conversely, it is quite frequent to observe small or medium-sized nicely meandering rivers discharging into a sinuous but not regularly meandering large river.

The shear force exerted by the flow on the river banks is not considered in this model, which means that no equation for bank erosion is included. However, it is implicitly assumed that the rate of bank erosion is directly proportional to the curvature of the flow. Eqs. (10), (14) and (16), which define the shape of the meander loops, all involve a shape parameter, k, which is a function of the maximum deflection angle (θ_0). This implies that there is an infinite number of solutions to these equations, depending on the value of parameter k. In practice, the value of k should be small when the river starts meandering in its newly formed floodplain. With time, the meanders expand laterally as the value of k increases, due to erosion of the river banks. As long as the rate of erosion of the river banks at any given point is directly proportional to the curvature at that point, the shape of the loop will comply with the present model.

River meanders are known to slowly migrate in a downstream direction. Downstream migration is possible when the rate of erosion is larger downstream of the apex of the loop than upstream and this asymmetry is believed to be caused by secondary currents. Nevertheless, it may be supposed that the present model would apply, if all ideal conditions are met, for short term or medium term – from a few days to a few months - modeling of meander loops but secondary currents are to be taken into account for long term modeling.

7 CONCLUSIONS

The model presented in this paper is based on a set of assumptions which may be approached on some river reaches, thereby resulting in regularly-shaped meander loops. In its simplest form, the model generates symmetrical meander loops having minimum variance. It also provides simple explicit equations defining meander path length, meander wave length and meander amplitude. These equations clearly demonstrate that the path length and wave length are not unique functions of river width but also depend on other river and flood plain characteristics. One of the parameters included in the model is the floodplain transverse slope which, apparently, has never been considered before in other meandering models. The model development for the general case results in the generation of upstream-skewed meander loops which are commonly observed in nature. It is shown that the upstream skew is due a non-negligible ratio of the longitudinal valley slope to the flood plain transverse slope,

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